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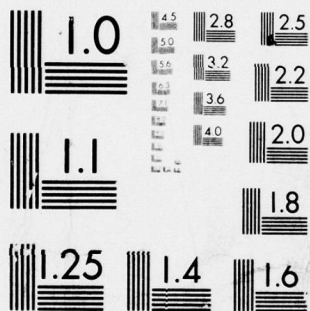
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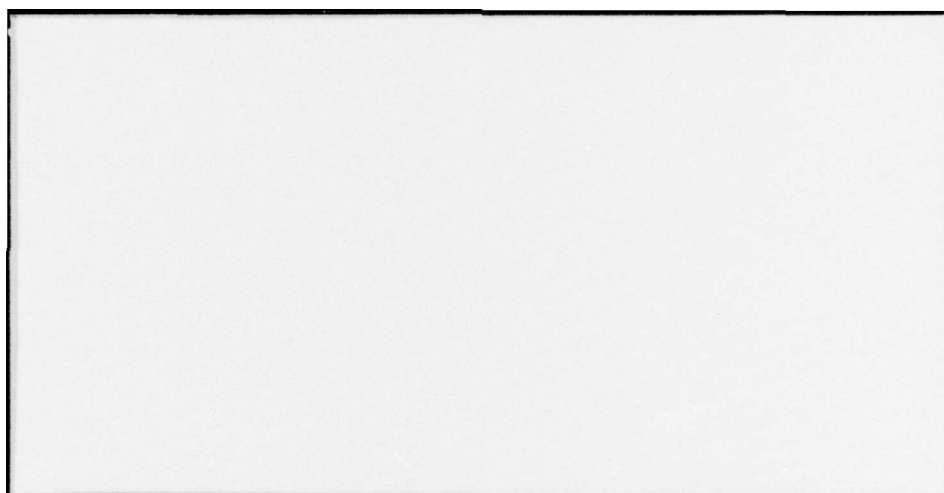
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ELECTROMAGNETIC SIGNAL PROPAGATION
IN CRYSTALS

Calvin H. Wilcox

Technical Summary Report #33

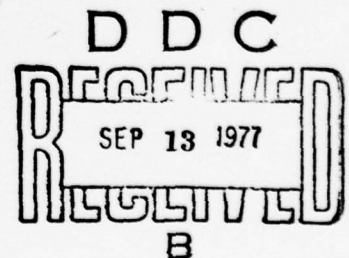
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Abstract

The propagation of the electromagnetic signals generated by localized electromagnetic current distributions in perfect crystals is analyzed under the hypotheses that dispersion and magnetic anisotropy are negligible and all wavelengths are large compared with interatomic distances. The signal fields are shown to converge asymptotically in energy, for $t \rightarrow \infty$, ^{approaching infinity} to the sum of two waves which propagate outward from the source region with the group velocities of the crystal. The polarizations and waveforms of these waves are calculated explicitly in terms of the crystal parameters and the source field.

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§1. Introduction.

This paper presents an analysis of the propagation of electromagnetic signals in perfect crystals under the following physical hypotheses:

- The sources of each signal are electric currents that are localized in a bounded portion of the crystal.
- The frequencies contained in the sources are limited to a band in which dispersion and magnetic anisotropy in the crystal are negligible.
- The wavelengths corresponding to the frequency band of the sources are large compared with the interatomic distances in the crystal.

Each such signal can be characterized by its electric field \vec{E} and magnetic field \vec{H} which satisfy Maxwell's equations with a tensor dielectric constant ϵ and a scalar magnetic permeability μ_0 that are characteristic of the crystal. Thus

$$(1.1) \quad \nabla \times \vec{H} - \epsilon \frac{\partial \vec{E}}{\partial t} = \vec{J}, \quad \nabla \times \vec{E} + \mu_0 \frac{\partial \vec{H}}{\partial t} = \vec{0}$$

where \vec{J} is the density of the electric currents that generate the signal. It is assumed that $\vec{J} = \vec{0}$ for all $t < t_0$ and the signal generated by \vec{J} is characterized as the solution of (1.1) that satisfies the initial condition

$$(1.2) \quad \vec{E} = \vec{0}, \quad \vec{H} = \vec{0} \text{ for all } t < t_0$$

A solution of the initial value problem (1.1), (1.2) was given by G. Herglotz [3]. An exposition of his method may be found in [2]. This solution makes it possible to calculate the signal fields \vec{E} and \vec{H} at any

point in space-time. However, it does not reveal the structure of the signal or its dependence on \vec{J} . The analysis presented below is based on the author's work on strongly propagative anisotropic media [8]. The main result of that work is a construction which assigns to each wave field with finite energy an asymptotic wave field which describes a wave that propagates outward with the group speeds of the medium. The difference between the exact and asymptotic wave fields tends to zero in energy when $t \rightarrow \infty$. The primary features of the asymptotic wave are determined by the medium. Superposed on these features is a waveform that is characteristic of the wave sources. In this paper the results of [8] are used to construct asymptotic wave fields for electromagnetic signals in crystals.

The remainder of the paper is organized as follows. §2 contains a summary of the properties of the slowness surfaces and wave surfaces for the Maxwell system that are needed in the subsequent analysis. In §3 a simple Hilbert space method is used to solve the initial value problem and derive the slowness surface representation of the signal field \vec{E} that is the starting point for the asymptotic analysis of [8]. In §4 the asymptotic wave fields for \vec{E} are constructed and their dependence of their polarization and waveform on the sources and crystal structure are determined.

§2. The Slowness and Wave Surfaces.

The properties of the slowness and wave surfaces of general strongly propagative media were discussed in [8]. Thorough discussions of the case of electromagnetic waves in crystals may be found in the books of A. Sommerfeld [6] and Courant-Hilbert [2] where the surfaces are called the normal and ray surfaces respectively. This section contains a review of the definitions and properties of these surfaces that are needed for the analysis of electromagnetic signal propagation.

If the field \vec{H} is eliminated between the two equations (1.1) the equation

$$(2.1) \quad \epsilon \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \nabla \times \nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{J}}{\partial t}$$

is obtained. To solve this equation it is convenient to use a system of Cartesian coordinates $x = (x_1, x_2, x_3) \in R^3$ with axes along the principal axes of the dielectric tensor ϵ . In such systems the components of ϵ have the diagonal form $\epsilon_{jk} = \epsilon_j \delta_{jk}$ where ϵ_1, ϵ_2 and ϵ_3 are positive. The notation $c_j = (\epsilon_j \mu_0)^{-1/2}$ will also be used. For definiteness it will be assumed that

$$(2.2) \quad 0 < c_1 \leq c_2 \leq c_3$$

Equation (2.1) is equivalent to the system

$$(2.3) \quad \frac{\partial^2 E_j}{\partial t^2} + c_j^2 \frac{\partial}{\partial x_j} \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} \right) - c_j^2 \Delta E_j = -\epsilon_j^{-1} \frac{\partial J_j}{\partial t}, \quad j = 1, 2, 3$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ and E_j and J_j are the components of \vec{E} and \vec{J} in the coordinate system (x_1, x_2, x_3) . The magnetic fields corresponding to solutions of (2.3) can be obtained from the second equation of (1.1) by a t -integration.

Plane waves in the crystal are solutions of (2.3) with $J_j = 0$ which have the form

$$(2.4) \quad E_j = e^{i(p \cdot x - \omega t)} a_j, \quad j = 1, 2, 3$$

where the a_j are independent of t and x . Substitution of (2.4) into (2.3) with $J_j = 0$ leads to three linear equations for a_1, a_2, a_3 which may be obtained by the substitutions $\partial/\partial t \rightarrow -i\omega$, $\partial/\partial x_j \rightarrow ip_j$, $E_j \rightarrow a_j$ and $J_j \rightarrow 0$. Non-trivial solutions exist if and only if the determinant of this system is zero. This condition reduces to the following dispersion relation between the frequency ω and the components p_j of the wave vector.

$$(2.5) \quad Q(\omega^2, p) \equiv \omega^2(\omega^4 - Q_2(p)\omega^2 + Q_4(p)) = 0$$

where

$$(2.6) \quad Q_2(p) = (c_2^2 + c_3^2) p_1^2 + (c_3^2 + c_1^2) p_2^2 + (c_1^2 + c_2^2) p_3^2$$

$$(2.7) \quad Q_4(p) = |p|^2 (c_2^2 c_3^2 p_1^2 + c_3^2 c_1^2 p_2^2 + c_1^2 c_2^2 p_3^2)$$

The equation $Q(\mu, p) = 0$ has the roots $0 \equiv \mu_0(p) \leq \mu_1(p) \leq \mu_2(p)$ where

$$(2.8) \quad \mu_1(p) = \frac{1}{2} \{Q_2(p) - \sqrt{D(p)}\}, \quad \mu_2(p) = \frac{1}{2} \{Q_2(p) + \sqrt{D(p)}\},$$

$$(2.9) \quad D(p) = Q_2(p)^2 - 4Q_4(p) = [(C_1 p_1 + C_3 p_3)^2 + C_2^2 p_2^2][(C_1 p_1 - C_3 p_3)^2 + C_2^2 p_2^2]$$

and $C_1, C_2, C_3 \geq 0$ are defined by

$$(2.10) \quad C_1^2 = c_3^2 - c_2^2, \quad C_2^2 = c_3^2 - c_1^2, \quad C_3^2 = c_2^2 - c_1^2$$

Hence the dispersion relation has the non-zero solutions

$$(2.11) \quad \omega = \lambda_{\alpha}(p) = \sqrt{\mu_{\alpha}(p)}, \quad \alpha = 1, 2$$

The functions $\lambda_{\alpha}(p)$ are homogeneous of degree 1. Hence to each $p \neq 0$ there correspond two plane waves (2.4) with phase speeds

$$(2.12) \quad \lambda_{\alpha}(p)/|p| = \lambda_{\alpha}(p/|p|), \quad \alpha = 1, 2$$

An electromagnetic classification of crystals may be based on the p -dependence of the phase speeds (2.12); cf. [6]. There are three classes which are defined by the conditions (1) $\epsilon_1 = \epsilon_2 = \epsilon_3$, (2) $\epsilon_1 > \epsilon_2 = \epsilon_3$ or $\epsilon_1 = \epsilon_2 > \epsilon_3$ and (3) $\epsilon_1 > \epsilon_2 > \epsilon_3$. Crystals in the first class are called isotropic because $\lambda_1(p) = \lambda_2(p) = c|p|$ and the phase speeds are independent of $p/|p|$. Crystals in the second class are called uniaxial because $\lambda_1(p)$ and $\lambda_2(p)$ coincide along a single axis. Finally, crystals in the third class are called biaxial because $\lambda_1(p)$ and $\lambda_2(p)$ coincide along the two distinct axes through the origin and the points $(\sqrt{c_2^2 - c_1^2}, 0, \pm\sqrt{c_3^2 - c_2^2})$ [6]. Note that by (2.9) $D(p)$ is the square of a polynomial if and only if at least one C_j is zero. Hence the biaxial crystals are precisely those for which $Q(\mu, p)$ is irreducible. In the remainder of the paper only the biaxial case is discussed. Results for the other cases can be obtained by allowing one or both of the constants ϵ_j to coincide.

The variation of the phase speeds (2.12) with $p/|p|$ can be visualized by means of the slowness surface $S = S_1 \cup S_2$ defined by

$$(2.13) \quad S_{\alpha} = \{p \in \mathbb{R}^3: \lambda_{\alpha}(p) = 1\} = \{p \in \mathbb{R}^3: |p| = \lambda_{\alpha}^{-1}(p/|p|)\}, \quad \alpha = 1, 2$$

The definition of the roots $\lambda_{\alpha}(p)$ implies that S is the real algebraic variety

$$(2.14) \quad S = \{p \in R^3: Q(1,p) = 0\}$$

It is a member of the class of slowness surfaces whose geometry was studied in [8]. Each S_α is a closed surface which is star-shaped with respect to the origin. Although $Q(1,p)$ is irreducible its restrictions to the three coordinate planes split into quadratic factors and the sections of S in each of these planes consist of a circle and an ellipse. It is clear from (2.8), (2.9) that the set $Z'_S = S_1 \cap S_2$ of singular points of S consists of the four points in which the two axes where $\lambda_1(p) = \lambda_2(p)$ intersect S . The two sets $S_\alpha - Z'_S$ are disjoint analytic surfaces. It is known that S is a Fresnel surface with equation [6]

$$(2.15) \quad S = \left\{ p \in R^3: \sum_{j=1}^3 \frac{p_j^2}{c_j^2 |p|^2 - 1} = 0 \right\}$$

Fresnel surfaces were studied extensively in the nineteenth and early twentieth centuries; see, e.g. [1, 5].

It will be shown in §4 that electromagnetic signals in crystals propagate outward from the source region with the group velocities defined by

$$(2.16) \quad x = \nabla_p \omega = \nabla_p \lambda_\alpha(p), \quad \alpha = 1, 2$$

The homogeneity of $\lambda_\alpha(p)$ and (2.13) imply that (2.16) defines a point x on the tangent plane $x \cdot p = 1$ to S_α at p . The variation of the group speeds $v_\alpha = |\nabla_p \lambda_\alpha|$ with the propagation direction $\theta = x/|x| = \nabla_p \lambda_\alpha / |\nabla_p \lambda_\alpha|$ can be visualized by means of the wave surface

$$(2.17) \quad W = \{x \in R^3: x \cdot p = 1 \text{ is a tangent plane to } S\}$$

W is the polar reciprocal of S [2, 8]. It is known that it is also a

Fresnel surface, namely the surface

$$(2.18) \quad W = \left\{ x \in \mathbb{R}^3: \sum_{j=1}^3 \frac{x_j^2}{c_j^{-2}|x|^2-1} = 0 \right\}$$

obtained from (2.15) by the substitutions $p \rightarrow x$, $c_j \rightarrow c_j^{-1}$. In particular, $W = W_1 \cup W_2$ where

$$(2.19) \quad W_\alpha = \{x \in \mathbb{R}^3: \lambda'_\alpha(x) = 1\}, \quad \alpha = 1, 2$$

and $\lambda'_\alpha(x)$ is obtained from $\lambda_\alpha(p)$ by the same substitutions. It follows that there are exactly two group velocities in each direction $\theta = x/|x|$ except the directions of the four singular points of W . The corresponding group speeds are given by

$$(2.20) \quad v_\alpha(\theta) = 1/\lambda'_\alpha(\theta), \quad \alpha = 1, 2$$

It is clear from (2.8) that $0 < v_2(\theta) \leq v_1(\theta)$ for all θ .

The Gauss map $N: S \rightarrow S^2 = \{\theta \in \mathbb{R}^3: |\theta| = 1\}$ and the polar reciprocal map $T: S \rightarrow W$ are also needed in §4. By definition, N assigns to each $p \in S$ the set of all exterior unit normal vectors to S at p ; see [8] for the construction of $N(p)$ at singular points. In the case of the Fresnel surface (2.15) N is single-valued and analytic except at the set Z'_S of four singular points. T is defined by

$$(2.21) \quad T(p) = (p \cdot N(p))^{-1} N(p), \quad p \in S$$

It is known that $p \cdot N(p)$ is bounded away from zero [8]. Hence T is also single-valued and analytic on $S - Z'_S$. To each $p \in Z'_S$ it assigns one of the four circles defined by the intersections of W with the four planes $\sqrt{c_2^2 - c_1^2} x_1 \pm \sqrt{c_3^2 - c_2^2} x_3 \pm c_2 \sqrt{c_3^2 - c_1^2} = 0$. The union of these circles will

be denoted by $Z_W'' = TZ_S'$. Note that $x = T(p)$ is given by (2.16) at the regular points of S . Since S is also the polar reciprocal of W the map $T^{-1}: W \rightarrow S$ has the same form as T ; that is, $T^{-1}(x) = (x \cdot N'(x))^{-1} N'(x)$ where $N': W \rightarrow S^2$ is the Gauss map of W . T^{-1} is single-valued and analytic on $W - Z_W'$ and $T^{-1}Z_W' = Z_S''$ is the set of four circles in which S intersects the four planes $\sqrt{c_1^{-2}-c_2^{-2}} p_1 \pm \sqrt{c_2^{-2}-c_3^{-2}} p_3 \pm c_2^{-1} \sqrt{c_1^{-2}-c_3^{-2}} = 0$. It was shown in [8] that if $Z_S = Z_S' \cup Z_S''$ and $Z_W = Z_W' \cup Z_W''$ then T is bijective and analytic between $S - Z_S$ and $W - Z_W$.

Finally, the inverse Gauss map $N^{-1}: S^2 \rightarrow S$ is needed in §4. It is clear from (2.21) and the geometry of W that N^{-1} is double-valued for all $\theta \in S^2 - Z_0$ where Z_0 is the intersection of S^2 with the four rays through the singular points of W . More precisely, if

$$(2.22) \quad x^{(\alpha)}(\theta) = v_{\alpha}(\theta) \theta, \quad \alpha = 1, 2$$

then

$$(2.23) \quad p^{(\alpha)}(\theta) = T^{-1} x^{(\alpha)}(\theta), \quad \alpha = 1, 2$$

defines the two branches of N^{-1} and each $p^{(\alpha)}$ is analytic on $S^2 - Z_0$.

§3. Representation of the Signals by Slowness Surface Integrals.

In this section Fourier analysis is used to represent the signals as slowness surface integrals. The analysis follows the method of [8, 9] to which reference is made for analytical details. Only the main steps are given here.

The Maxwell equations (2.3) may be written as a matrix system

$$(3.1) \quad \frac{\partial^2 u}{\partial t^2} + A u = f(t, x)$$

where $u = u(t, x) = (E_1, E_2, E_3)^T$ (T denotes the transpose), $f = -\epsilon^{-1} \partial J / \partial t$, $\epsilon = \text{diag} (\epsilon_1, \epsilon_2, \epsilon_3)$ and $J = (J_1, J_2, J_3)^T$. A is the 3×3 matrix differential operator defined by $A = c^2 A_0$, $c^2 = \text{diag} (c_1^2, c_2^2, c_3^2)$ and $A_0 = (\partial^2 / \partial x_j \partial x_k - \delta_{jk} \Delta)$. The solution of the initial value problem for (3.1) will be constructed in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3, \mathbb{C}^3)$ with scalar product

$$(3.2) \quad (u, v)_{\mathcal{H}} = \int_{\mathbb{R}^3} u(x)^* c^{-2} v(x) dx$$

where $u^* = \bar{u}^T$ is the Hermitian adjoint of u . If $u \in \mathcal{H}$ then Au is well-defined as a distribution and a linear operator $A: \mathcal{H} \rightarrow \mathcal{H}$ is defined by

$$(3.3) \quad D(A) = \mathcal{H} \cap \{u: Au \in \mathcal{H}\}, \quad Au = Au$$

It can be shown by Fourier analysis that A is selfadjoint and non-negative: $A^* = A \geq 0$.

System (3.1) will be interpreted as an ordinary differential equation

$$(3.4) \quad \frac{d^2 u}{dt^2} + Au = f(t, \cdot), \quad t \in \mathbb{R}$$

for a function $t \rightarrow u(t, \cdot) \in \mathcal{H}$. To integrate (3.4) let $A^{1/2}$ be the

non-negative square root of A and note that $D(A^{1/2})$ is a Hilbert space under the graph norm of $A^{1/2}$. If it is assumed that the source function $J \in C^1(\mathbb{R}, D(A^{1/2}))$ and $\text{supp } J \subset \{t: t_0 \leq t \leq t_0 + T\}$ then $f = -\epsilon^{-1} dJ/dt \in C(\mathbb{R}, D(A^{1/2}))$, $\text{supp } f \subset \{t: t_0 \leq t \leq t_0 + T\}$ and

$$(3.5) \quad \int_{t_0}^{t_0+T} f(t, \cdot) dt = 0$$

The signal generated by f is by definition the solution of (3.4) that satisfies $u(t, \cdot) = 0$ for $t < t_0$. It is given by the Duhamel integral

$$(3.6) \quad u(t, \cdot) = \int_{t_0}^t \{\cos(t - \tau) A^{1/2}\} \int_{t_0}^T f(t', \cdot) dt' d\tau, \quad t \geq t_0$$

Indeed, the hypotheses on f and the spectral theorem imply that (3.6) defines a function $u \in C^2(\mathbb{R}, \mathcal{H}) \cap C^1(\mathbb{R}, D(A^{1/2})) \cap C(\mathbb{R}, D(A))$ that satisfies (3.4). Note that since J and f have real-valued components, $u(t, x) = \text{Re } \{v(t, x)\}$ where

$$(3.7) \quad v(t, \cdot) = e^{-itA^{1/2}} \int_{t_0}^t e^{i\tau A^{1/2}} \int_{t_0}^T f(t', \cdot) dt' d\tau$$

Moreover, (3.5) implies that for all $t \geq t_0 + T$

$$(3.8) \quad v(t, \cdot) = e^{-itA^{1/2}} h, \quad h = \int_{t_0}^{t_0+T} e^{i\tau A^{1/2}} \int_{t_0}^T f(t', \cdot) dt' d\tau$$

The wave function (3.8) will be constructed by Fourier analysis. To this end let $\Phi: \mathcal{H} \rightarrow \mathcal{H}$ denote the unitary operator defined by the Fourier transform. Then $A = \Phi^* A(\cdot) \Phi$ where $A(\cdot)$ denotes the matrix multiplication operator defined by $A(p) = c^2 A_0(p)$, $A_0(p) = (|p|^2 \delta_{jk} - p_j p_k)$,

$p \in \mathbb{R}^3$. To complete the spectral analysis of $A(p)$ note that its characteristic polynomial is the polynomial $Q(\mu, p)$ with roots $\mu_0(p) \equiv 0, \mu_1(p), \mu_2(p)$ defined by (2.8). Thus

$$(3.9) \quad (\mu I - A(p))^{-1} = \sum_{\alpha=0}^2 (\mu - \mu_{\alpha}(p))^{-1} \hat{P}_{\alpha}(p)$$

where the matrices $\hat{P}_{\alpha}(p)$ are given by

$$(3.10) \quad \hat{P}_{\alpha}(p) = \frac{\text{cof}(\mu I - A(p))}{\partial Q(\mu, p) / \partial \mu} \Big|_{\mu=\mu_{\alpha}(p)}, \quad \alpha = 0, 1, 2$$

and $\text{cof } M$ denotes the cofactor matrix of M . It follows as in [8] that $v(t, \cdot) = \exp(-itA^{1/2})h = \sum_{\alpha=0}^2 v_{\alpha}(t, \cdot)$ where

$$(3.11) \quad v_{\alpha}(t, x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i(x \cdot p - t\lambda_{\alpha}(p))} \hat{P}_{\alpha}(p) \hat{h}(p) dp$$

$\lambda_{\alpha}(p) = \sqrt{\mu_{\alpha}(p)}$ as in §2 and $\hat{h} = \Phi h$. In particular, $v_0(t, x) = v_0(x)$ is the static component of $v(t, x)$. The remaining two integrals may be combined as in [8, §4]. The result is the representation

$$(3.12) \quad v(t, x) = v_0(x) + \int_0^{\infty} e^{-it\lambda} w(\lambda x, \lambda) \lambda^2 d\lambda$$

where

$$(3.13) \quad w(x, \lambda) = (2\pi)^{-3/2} \int_S e^{ix \cdot p} \hat{P}(p) \hat{h}(\lambda p) |T(p)|^{-1} dS$$

is an integral over the slowness surface $S = S_1 \cup S_2$, $\hat{P}(p) = \hat{P}_{\alpha}(p)$ for $p \in S_{\alpha}$ and T is the polar reciprocal map of §2. This representation is the starting point for the analysis in [8] of the asymptotic behavior of $v(t, \cdot)$ in \mathcal{H} .

§4. Electromagnetic Signal Structure in Crystals.

Consider a source function $J(t, x)$ which is localized in both space and time:

$$(4.1) \quad \text{supp } J \subset \{(t, x): t_0 \leq t \leq t_0 + T \text{ and } |x - x_0| \leq \delta_0\}$$

The signal generated by J will reach points x in the far field of the sources, characterized by $|x - x_0| \gg \delta_0$, after a time interval of magnitude comparable with $c_1 |x - x_0|$. Hence the far field form of $u(t, x)$ coincides with its form for large t . The latter is given by the asymptotic wave functions of [8].

The analysis of [8], applied to (3.12), (3.13), implies that the signal $u(t, x) = \text{Re } \{v(t, x)\}$ generated by $f = -\epsilon^{-1} \partial J / \partial t$ satisfies

$$(4.2) \quad \lim_{t \rightarrow \infty} \|u(t, \cdot) - u_0 - u^\infty(t, \cdot)\|_{\mathcal{H}} = 0$$

where $u_0(x) = \text{Re } \{v_0(x)\}$ is the static component of the signal, defined in §3, and $u^\infty(t, x)$ is the asymptotic wave function for u . The latter has the form

$$(4.3) \quad u^\infty(t, x) = |x|^{-1} \sum_{\alpha=1}^2 G(x \cdot p^{(\alpha)}(\theta) - t, p^{(\alpha)}(\theta)), \quad x = |x|\theta$$

where $p^{(\alpha)}$, $\alpha = 1, 2$, are the branches of the inverse Gauss map of S defined in §2. Note that $T p^{(\alpha)}(\theta) = v^{(\alpha)}(\theta)\theta$, by (2.22), (2.23), and $p \cdot T(p) = 1$ for all $p \in S$. Hence

$$(4.4) \quad x \cdot p^{(\alpha)}(\theta) = |x|\theta \cdot p^{(\alpha)}(\theta) = |x|/v_\alpha(\theta)$$

and (4.3) represents the sum of two waves which propagate outward with the group speeds $v_\alpha(\theta)$. The form (4.3) of the wave is determined solely

by the crystal. The shape of the wave profile $G: R \times S \rightarrow R^3$ depends on both the crystal structure and the source function f . It is given by $G(\tau, p) = \text{Re} \{F(\tau, p)\}$ where $F: R \times S \rightarrow C^3$ is defined by

$$(4.5) \quad F(\tau, p) = \Pi(p) \left[(2\pi)^{-1/2} \psi(p) \int_0^\infty e^{i\tau\lambda} \hat{h}(\lambda p) \lambda d\lambda \right]$$

and

$$(4.6) \quad \Pi(p) = |K(p)|^{-1/2} |T(p)|^{-1} \hat{P}(p)$$

where $K(p)$ is the Gaussian curvature of S at p , $\psi(p) = \exp\{i\frac{\pi}{4}(v^+(p) - v^-(p))\}$ and $v^+(p)$ and $v^-(p)$ denote the number of principal curvatures of S at p that are positive and negative, respectively. Note that $\Pi(p)$ and $\psi(p)$ are defined and analytic for all $p \in S - Z_S$. The integral in (4.4) converges in the Hilbert space $\mathcal{H}(S)$ with norm defined by

$$(4.7) \quad \|F\|_{\mathcal{H}(S)}^2 = \int_0^\infty \int_S F(\tau, p)^* c^{-2} F(\tau, p) |K(p) T(p)| dS d\tau$$

and the map $\Theta: \mathcal{H} \rightarrow \mathcal{H}(S)$ defined by $\Theta h = F$ is an isometry [8].

The profile (4.5) corresponding to the signal (3.8) can now be calculated. Note that

$$(4.8) \quad \hat{h}(p) = \Phi h(p) = \int_{t_0}^{t_0+T} \exp\{i\tau A(p)^{1/2}\} \int_{t_0}^T \Phi f(t, p) dt d\tau$$

Moreover, the definition of $\hat{P}(p)$ implies that $\hat{P}(p)\Psi(A(p)) = \Psi(1)\hat{P}(p)$ for all $p \in S - Z_S$. Thus if $p \in S - Z_S$, $\lambda > 0$ then

$$(4.9) \quad \hat{P}(p) \hat{h}(\lambda p) = \hat{P}(p) \int_{t_0}^{t_0+T} e^{i\lambda t} \int_{t_0}^T \Phi f(t, p) dt d\tau$$

Finally, integration by parts with respect to τ and use of (3.5) gives

$$(4.10) \quad \hat{P}(p) \hat{h}(\lambda p) = (2\pi)^{1/2} i \lambda^{-1} \hat{P}(p) \hat{f}(-\lambda, \lambda p), \quad p \in S - Z_S$$

where $\hat{f}(\omega, p)$ denotes the four-dimensional Fourier transform of $f(t, x)$.

Thus, since $\Pi(p)$ has real components, the profile $G(\tau, p) = \text{Re} \{F(\tau, p)\}$ generated by f has the form

$$(4.11) \quad G(\tau, p) = \Pi(p) \text{Re} \left\{ i\psi(p) \int_0^\infty e^{i\tau\lambda} \hat{f}(-\lambda, \lambda p) d\lambda \right\}$$

Note that, since $\hat{f}(\omega, p)$ is the Fourier transform of a real-valued function

$$(4.12) \quad \int_0^\infty e^{i\tau\lambda} \hat{f}(-\lambda, \lambda p) d\lambda = s_f(-\tau, p) - i \sigma_f(-\tau, p)$$

where $s_f(\tau, p)$ and $\sigma_f(\tau, p)$ are a pair of Hilbert transforms with respect to τ [7]. Moreover, an application of Fubini's theorem gives

$$(4.13) \quad s_f(\tau, p) = \frac{1}{2} \int_{-\infty}^\infty e^{-i\tau\lambda} \hat{f}(-\lambda, \lambda p) d\lambda = \frac{1}{4\pi} \int_{\mathbb{R}^3} f(p \cdot x + \tau, x) dx$$

Hence the signal waveform $s_f(\tau, p)$ is the integral of $f(t, x)$ over the space-time hyperplane $t = p \cdot x + \tau$. Thus $s_f(\tau, p)$ is essentially the four-dimensional Radon transform of $f(t, x)$ [4].

Finally, note that the phase factor $\psi(p)$ in (4.10) satisfies $\psi(p) = -i$ at elliptic points of S and $\psi(p) = 1$ at hyperbolic points. Moreover S_2 , the inner sheet of S , is convex and has only elliptic points while S_1 , the outer sheet, is divided into five components by the four circles which make up the set Z_S'' of parabolic points. The union of the four components that contain singular points is the set S^- of hyperbolic points of S and $S = S^+ \cup S^- \cup Z_S$, a disjoint union. Combining (4.11),

(4.12), (4.13) and these remarks gives the signal waveform

$$(4.14) \quad G(\tau, p) = \begin{cases} \Pi(p) s_f(-\tau, p), & p \in S^+ \\ \Pi(p) \sigma_f(-\tau, p), & p \in S^- \end{cases}$$

where $s_f(\tau, p)$, $\sigma_f(\tau, p)$ is the Hilbert transform pair defined by (4.12).

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